A Task-Driven Approach to Unified Synthesis of Planar Four-Bar Linkages Using Algebraic Fitting of a Pencil of G-Manifolds

This paper studies the problem of planar four-bar motion generation from the viewpoint of extraction of geometric constraints from a given set of planar displacements. Using the image space of planar displacements, we obtain a class of quadrics, called generalized- or G-manifolds, with eight linear and homogeneous coefficients as a unified representation for constraint manifolds of all four types of planar dyads, RR, PR, and PP. Given a set of image points that represent planar displacements, the problem of synthesizing a planar four-bar linkage is reduced to finding a pencil of G-manifolds that best fit the image points in the least squares sense. This least squares problem is solved using singular value decomposition (SVD). The linear coefficients associated with the smallest singular values are used to define a pencil of quadrics. Additional constraints on the linear coefficients are then imposed to obtain a planar four-bar linkage that best guides the coupler through the given displacements. The result is an efficient and linear algorithm that naturally extracts the geometric constraints of a motion and leads directly to the type and dimensions of a mechanism for motion generation. [DOI: 10.1115/1.4035528]

1 Introduction

In this paper, we present a task-driven approach to unified type and dimensional synthesis of planar four-bar linkage mechanisms. Planar linkages are the most common form of mechanisms found in mechanical systems and have been a subject of interest and research in machine design area for many decades. Some key texts that describe state of the art as well as established methods and theory in kinematic synthesis of machines are by McCarthy and Soh [1], Sandor and Erdman [2], Hunt [3], Hartenberg and Denavit [4], and Suh and Radcliffe [5]. Despite having been a research topic alive for a long time, various proposed solutions to planar mechanism design for the approximate motion synthesis have been nonlinear in nature. In general, the algorithms proposed are computationally expensive and require dealing with the type and dimension synthesis separately. In this paper, we follow Wu et al. [6] and study the problem of planar motion approximation from the viewpoint of kinematic extraction of geometric constraints from a given set of planar displacements. Using a kinematic mapping of planar kinematics, we propose a general algebraic method for unified type and dimensional synthesis of planar four-bar linkages, which reveals the geometric constraints implicit in the given motion via a linear, two-step process. The method is fast, efficient, and provides type and dimensions of the mechanisms, which can execute that motion. This paper is an extension of our earlier work on dimensional synthesis of planar 4R linkages [7], wherein our focus was only on the motions that could be executed by RR dyads. The main contributions of this work are in (1) presenting a unified representation for the motion of all possible planar dyads, and (2) devising a simple linear method for naturally extracting the constraints hidden in a given motion and matching it with a four-bar motion without presumption of the type of a linkage. In addition, we also show via an example as to how the approach for four-bar linkage synthesis can be applied to six-bar linkages as well.

The earliest approach to the motion synthesis problem was dealt with by Burmester [8], who posited that a given four-bar linkage can go through at most five positions exactly (precision position synthesis). For a continuous motion or more than five positions, typically only an approximate motion synthesis can be performed. For this problem, Ravani and Roth [9,10] proposed a kinematic mapping approach. Blaschke [11] and Grunwald [12] had given rise to the concept of kinematic mapping almost a century ago, but it did not find many practical applications until the work of Ravani and Roth. A modern treatment of kinematic mapping can be found in the formative texts of Bottema and Roth [13] and McCarthy [14]. In the kinematic mapping approach to synthesis, planar displacements in Cartesian space are mapped into points in a three-dimensional projective space (called image space of planar kinematics), while workspace constraints of a mechanism map into algebraic manifolds (called constraint manifold) in the same space. In this way, a single degree-of-freedom (DOF) motion of a planar mechanism is represented by the intersection curve of two algebraic surfaces in the image space. The problem of motion approximation is transformed into an algebraic curve fitting problem in the image space, where various methods in approximation theory may be applied. This includes the definition of the approximation error (called structural error) in the image space, formulation of a least squares problem and application of appropriate numerical methods to find values of the design variables for minimization of the error. Pursuant to Ravani and Roth’s kinematic mapping approach for mechanism synthesis, further research has been done by Bodduluri and McCarthy [15], Bodduluri [16], Larcheulle [17-18], Ge and Larcheulle [19], Husty et al. [20], and more recently by Wu et al. [21], Purwar and Gupta [22], and Hayes et al. [23,24]. Schrcker et al. [25] applied the kinematic mapping approach to detect branch defect in the planar four-bar linkage synthesis—a result that can be used in this work as well.

In this paper, we are dealing with the use of the image space of planar kinematics for approximate task-driven simultaneous type and dimensional synthesis of planar four-bar linkages. While the...
constraint manifolds associated with planar four-bar linkages are algebraic, geometric (or, normal) distances have been used as default metric for least-squares fitting of these algebraic manifolds. Ravani and Roth [9] used normal distance to develop a least-squares algorithm for fitting the image curve of a four-bar motion. Their algorithm has two features: (1) fit the set of image points to two constraint manifolds simultaneously; and (2) use a tangent hyperplane approximation of constraint manifolds to obtain the normal distance. The resulting algorithm is highly non-linear and requires many initial choices to converge to a reasonable solution. Larochelle [26,27] presented a different approach to linear and requires many initial choices to converge to a reason-

obtain the normal distance. The resulting algorithm is highly non-

tangent hyperplane approximation of constraint manifolds to

least-squares algorithm for fitting the image curve of a four-bar

body by an angle \( \phi \). Let \( M \) denote a coordinate frame attached to

\( \begin{bmatrix} \cos \phi & -\sin \phi & d_1 \\ \sin \phi & \cos \phi & d_2 \\ 0 & 0 & 1 \end{bmatrix} \)

(1)

The line coordinate transformation for the same displacement is given by the transpose of the inverse of \([H]\) (see Ref. [13])

\[
[H] = \left( [H]^{-1} \right)^T =
\begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
-d_1 \cos \phi - d_2 \sin \phi & d_1 \sin \phi - d_2 \cos \phi & 1
\end{bmatrix}
\]

(2)

The transformations \([H]\) and \([H]\) are said to be dual to each other. Introducing the following mapping from Cartesian space parameters \((d_1, d_2, \phi)\) to image space coordinates \( Z = (Z_1, Z_2, Z_3, Z_4) \) (see Ref. [9])

\[
Z_1 = \frac{1}{2} \left( d_1 \cos \phi + d_2 \sin \phi \right), \quad Z_2 = \frac{1}{2} \left( -d_1 \sin \phi + d_2 \cos \phi \right),
\]

\[
Z_3 = \sin \phi, \quad Z_4 = \cos \phi
\]

(3)

we can reparameterize the homogeneous transforms \([H]\) and \([H]\) in quadratic form

\[
[H] = \begin{bmatrix}
Z_1^2 - Z_3^2 & -2Z_1Z_2 & 2(Z_2Z_3 + Z_2Z_4) \\
2Z_1Z_2 & Z_2^2 - Z_3^2 & 2(Z_2Z_3 + Z_2Z_4) \\
0 & 0 & Z_3^2 + Z_4^2
\end{bmatrix}
\]

(4)

\[
[H] = \begin{bmatrix}
Z_1^2 - Z_3^2 & -2Z_1Z_2 & 0 \\
2Z_1Z_2 & Z_2^2 - Z_3^2 & 0 \\
2(Z_2Z_3 - Z_2Z_4) & 2(Z_2Z_3 + Z_2Z_4) & Z_3^2 + Z_4^2
\end{bmatrix}
\]

(5)

where \( Z_1^2 + Z_3^2 = 1 \).

Equation (3) defines a mapping from the Cartesian space parameters \((d_1, d_2, \phi)\) to a three-dimensional (3D) projective quasi-elliptic space parameterized by the homogeneous coordinates of the point \( Z \). This is called the kinematic mapping of planar displacements and the corresponding 3D projective space is called the Image Space of planar displacement, denoted as \( \Sigma \). There is no real planar displacement that maps to the points on the real line given by \( Z_3 = Z_4 = 0 \). Thus, a planar displacement is represented by a point in \( \Sigma \); a single degree-of-freedom (DOF) motion is represented by a curve; and a two DOF motion is represented by a surface in \( \Sigma \). For details on kinematic mapping and the properties of the image space, see Refs. [9] and [13].

### 3 Constraining a Planar Displacement

In this paper, we consider only one- and two-DOF motions that are constrained by simple geometric constraints such as lines and circles. This includes 2DOF planar motions of a rigid body subject to one of the following four types of geometric constraints:

1. one of its points stays on a circle: this can be realized by a planar \( RR \) dyad, where \( R \) denotes a revolute joint; see Fig. 1(a)

2. Parameterizing a Planar Displacement

A planar displacement can be decomposed into the translation of a point \((d_1, d_2)\) on the moving body as well as rotation of the body by an angle \( \phi \). Let \( M \) denote a coordinate frame attached to the moving body and \( F \) be a fixed reference frame. Then, a planar displacement can be represented as a transformation of point or line coordinates from \( M \) to \( F \). The point coordinate homogeneous transformation matrix associated with a planar displacement is given by

\[
\begin{bmatrix}
\cos \phi & -\sin \phi & d_1 \\
\sin \phi & \cos \phi & d_2 \\
0 & 0 & 1
\end{bmatrix}
\]

(1)
(2) one of its points stays on a line: this can be realized by a planar PR dyad, where P denotes a prismatic joint; see Fig. 1(b).

(3) one of its lines stays tangent to a given circle: this can be realized by a planar RP dyad; see Fig. 1(c).

(4) one of its lines translates along another line: this can be realized by a planar PP dyad; see Fig. 1(d).

A planar motion subject to any two constraints listed above (including two of the same types) results in a 1DOF motion called planar four-bar motion. Planar four-bar linkages include planar 4R, slider–crank, inversions of slider–crank, as well as double-slider mechanisms; see Fig. 2 for some such linkages. In this section, we develop representations of circular and linear constraints that lead to a unified representation of planar dyad motions listed above.

Let $\mathbf{X} = (x_1, x_2, x_3)$ (where $x_3 \neq 0$) denote the homogeneous coordinates of a point. Then the following equation:

$$2a_1x_1 + 2a_2x_2 + a_3x_3 = a_0 \left( \frac{x_1^2 + x_2^2}{x_3} \right)$$

(6)

defines a circle $C$ when $a_0 \neq 0$. The center of the circle is given by the homogeneous coordinates

$$a = (a_1, a_2, a_0)$$

(7)

and the radius $r$ of the circle satisfies

$$a_0^2r^2 - a_0a_3 = a_1^2 + a_2^2$$

(8)

When $a_0 = 0$, Eq. (6) reduces to the equation of a line

$$2a_1x_1 + 2a_2x_2 + a_3x_3 = 0$$

(9)

Thus, Eq. (6) is a unified presentation for both a circle and a line in the homogeneous form.

As a planar RR dyad and a PR dyad define, respectively, a 2DOF motion of a rigid body for which one of its points stays on a circle and on a line, Eq. (6) also provides a unified representation of geometric constraints associated with such two dyads.

We now consider an RP dyad that defines a 2DOF planar motion for which one of its lines stays tangent to a given circle $C$. This requires a line representation of a circle. First, we recast Eq. (6) in matrix form

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -a_0 & 0 & a_1 \\ 0 & -a_0 & a_2 \\ a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

(10)

The adjoint of the coefficient matrix in above is given by

$$[C_{adj}] = \begin{bmatrix} -a_0a_3 - a_2^2 & a_1a_3 & a_0a_1 \\ a_1a_2 & -a_0a_3 - a_1^2 & a_0a_2 \\ a_0a_1 & a_0a_2 & a_0^2 \end{bmatrix}$$

(11)

It is well known in projective geometry of conics (see Ref. [29]) that a line with coordinates $\mathbf{L} = (L_1, L_2, L_3)$ stays tangent to the circle $C$ with center and radius given by Eqs. (7) and (8), respectively, when

$$\mathbf{L}^T [C_{adj}] \mathbf{L} = 0$$

(12)

Using Eq. (8), we can decompose $[C_{adj}]$ as

$$[C_{adj}] = \begin{bmatrix} a_0^2 & a_1a_2 & a_0a_1 \\ a_1a_2 & a_2^2 + a_0^2 & a_0a_2 \\ a_0a_1 & a_0a_2 & a_0^2 \end{bmatrix} - \begin{bmatrix} a_0^2r^2 & 0 & 0 \\ 0 & a_0^2r^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(13)

Substituting $[C_{adj}]$ from Eq. (13) into Eq. (12), we obtain, after some algebra

$$a_1L_1 + a_2L_2 + a_0L_3 = \pm a_0r \sqrt{L_1^2 + L_2^2}$$

(14)

This yields two lines that are $r$-distance away from the center of the circle, $a = (a_1, a_2, a_0)$. In addition, when $r = 0$, the two lines overlap into one that passes through $a$. This is usually the case with a swinging-block type dyad.

$PP$ dyad is a special type of dyad, whose second link actually follows a rectilinear motion with no change in orientation. The motion of a $PP$ dyad is constrained such that the angle between a line $\mathbf{L} = (L_1, L_2, L_3)$ and another line $(2a_1, 2a_2, 3)\mathbf{a}$ in $F$ is a constant, which can be described as

$$2a_1L_1 + 2a_2L_2 = k$$

(15)

where $k$ is a constant that corresponds to the angle between the two lines. Equation (15) can be seen as a special case of Eq. (14).

Thus, all the four planar dyads, RR, PR, RP, and PP, can be represented in terms of geometric constraints given by Eqs. (6) and (14). Furthermore, the left-hand side of Eqs. (6) and (14) is a linear combination of point and line coordinates, respectively.

4 A Unifying Representation for Planar Dyad Motions

In this section, we first derive algebraic form of a generalized quadric manifold that is common to 2DOF motions subject to the constraints containing linear and quadratic terms in Eqs. (6) and (14). We then show how this manifold can be used to develop a unified representation for constraint manifolds of planar RR, PR, RP, and PP dyads.

4.1 G-Manifolds for Planar Dyad Motions. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{X} = (X_1, X_2, X_3)$ denote the homogeneous coordinates of a point in the moving frame $M$ and the fixed frame $F$, respectively; and let $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{L} = (L_1, L_2, L_3)$ denote the homogeneous coordinates of a line in $M$ and $F$, respectively, where $l_1^2 + l_2^2 = 1$ and the absolute value of $L_1$ is the distance to the line from the origin of $M$. An algebraic form of the constraints of the dyads parameterized by image space coordinates can be obtained by substituting the fixed frame coordinates obtained from $\mathbf{X} = [H]\mathbf{x}$ or $\mathbf{L} = [H]\mathbf{l}$ in Eq. (6) or (14).
In view of $X = [H]x$, where $[H]$ is given by Eq. (4), a linear combination of the point coordinates as shown in point constraint Eq. (6) involves only five distinct elements of the matrix $[H]$; likewise, in view of $L = [H]l$, where $[H]$ is given by Eq. (5), a linear combination of the line coordinates as shown in Eq. (14) involves only five distinct elements of the matrix $[H]$. Furthermore, it can be shown that the nonlinear term $X_i^2 + X_j^2$ in Eq. (16) produces only one new element $(Z_i + Z_j)$. Thus, by collecting all of these independent terms appearing in the constraint equations, we obtain the following common representation of geometric constraints expressed by Eqs. (6) and (14) in terms of image space coordinates $Z_i$ ($i = 1, 2, 3, 4$):

$$q_1(Z_i^2 + Z_j^2) + q_2(Z_iZ_j - Z_iZ_j) + q_3(Z_iZ_j + Z_iZ_j) + q_4(Z_iZ_j - Z_iZ_j) + q_5(Z_iZ_j) + q_6(Z_iZ_j) = 0$$

(17)

This defines a quadric surface in the image space with eight homogeneous coefficients $q_i, i = 1, 2, \ldots, 8$. In this paper, we call this quadric a generalized constraint manifold, or G-manifold in short. For this generalized-manifold to become the constraint manifold (or C-manifold), of a planar RR, PR, RP, or PP dyad, one must impose additional constraints on the coefficients $q_i$.

4.2 C-Manifolds of RR and PR Dyads. Consider first a planar 2DOF motion of a rigid body for which a point $x$ on moving body remains on a circle with center $(a_1, a_2, a_3)$ and radius $r$ of the fixed frame, i.e., satisfies the circular constraint of Eq. (6). Substituting $X = [H]x$, where $[H]$ is given in Eq. (4), into Eq. (6), we obtain, after some algebra

$$2a_1(Z_iZ_j + Z_jZ_i) + 2a_2(Z_iZ_j - Z_iZ_j)$$

$$+ \left( -2a_1x_3 + 2a_2x_3 \right) Z_iZ_j + \left( a_1x_1 + a_2x_2 \right) (Z_i^2 - Z_j^2)$$

$$+ \left( 2a_1x_1 + 2a_2x_2 \right) Z_2Z_4$$

$$- 2a_1x_1(Z_iZ_j - Z_iZ_j) + 2a_2x_2x_3(Z_i^2 - Z_j^2)$$

$$+ \left( a_1x_1 - a_2x_2 \right) (Z_i^2 + Z_j^2) = 0$$

(18)

After collecting like terms, we obtain

$$-2a_1x_1(Z_i^2 + Z_j^2) + 2a_2x_3(Z_iZ_j - Z_iZ_j) + 2a_2x_3(Z_iZ_j - Z_iZ_j) + 2a_2x_3x_3(Z_i^2 - Z_j^2) + 2a_2x_2x_3(Z_i^2 - Z_j^2)$$

$$+ (a_1x_1 + a_2x_2)(Z_i^2 - Z_j^2) + \left( a_1x_1 - a_2x_2 \right) (Z_i^2 + Z_j^2) = 0$$

(19)

We may rewrite Eq. (19) in the form of G-manifolds (Eq. (17) with the following coefficients $q_i$:

$$q_1 = -2a_0x_3$$

$$q_2 = 2a_0x_1$$

$$q_3 = 2a_0x_2$$

$$q_4 = 2a_0x_3$$

$$q_5 = 2a_0x_3$$

$$q_6 = 2a_0x_3$$

$$q_7 = 2a_0x_3$$

$$q_8 = 2a_0x_3$$

(20)

It follows from Eq. (20) that the coefficients $q_i$ must satisfy the following two relations:

$$q_1q_6 + q_2q_3 - q_3q_4 = 0$$

$$2q_1q_7 - q_2q_4 - q_3q_5 = 0$$

(21)

Also $e$ is called the constraint fitting error, which can be used to show if a vector $p$ is qualified to represent a dyad.

4.3 C-Manifold of an RP Dyad. The motion of an RP dyad is constrained such that a line $l = (l_1, l_2, l_3)$ on moving body stays tangent to a given circle $C$ of fixed frame. Substituting $L = [H]l$, where $[H]$ is given by Eq. (5), and Eq. (16) into Eq. (14), we can put the resulting C-manifold in the same form as given by Eq. (17) where

$$q_1 = 0$$

$$q_2 = 2a_0d_1$$

$$q_3 = 2a_0d_2$$

$$q_4 = 0$$

$$q_5 = 0$$

$$q_6 = -2a_0l_1 + 2a_0l_2$$

$$q_7 = -a_0l_1 - a_0l_2$$

$$q_8 = a_0(l_3 + r)$$

(23)

As both $l_1$ and $r$ are lumped into $q_8$, without any loss of generality, we may set $r = 0$, i.e., requiring that the line $L$ passes through the fixed point $(a_1, a_2, a_3)$ instead of being tangent to the circle $C$. The set of five nonzero coefficients $(q_2, q_3, q_4, q_7, q_8)$ is homogeneous but otherwise independent of each other. Furthermore, since $q_1 = q_3 = q_5 = 0$, it follows that Eq. (21) is automatically satisfied. Projecting this manifold onto $Z_4 = 1$, one obtains a
hyperbolic paraboloid, the same type of quadric as obtained in case of a PR dyad.

4.4 C-Manifold of PP Dyad. PP dyad is a special type of mechanism, whose second link actually follows a translational motion with no change in orientation. The motion of a PP dyad is constrained such that the angle between a line \( l = (l_1, l_2, l_3) \) in \( M \) and a line \( (2a_1, 2a_2, a_3) \) in \( F \) is a constant. Substituting \( L = [H] \) in Eq. (15) as in an RP case, we can put the resulting constraint manifold in the same form as Eq. (17) where

\[
\begin{align*}
q_1 &= 0, \\
q_2 &= 0, \\
q_3 &= 0, \\
q_4 &= 2a_1l_1 - 2a_2l_1, \\
q_5 &= a_1l_1 + a_2l_2, \\
q_6 &= -\frac{k}{2}
\end{align*}
\]

Since \( q_1 \) through \( q_6 \) are all equal to zero, it follows that Eq. (21) is automatically satisfied. Projecting this manifold onto \( Z_1 = 1 \), one obtains two parallel planes in the form of \( Z_3 = \text{constant} \). With only two equations and four unknowns to solve for, the inverse computation will result in infinite solutions. This is because the position of the line can be arbitrary for pure translation.

4.5 Inverse Computations of Dyad Parameters. A unified form of the inverse computation relationships for RR, PR or RP type of dyad is given as follows:

\[
\begin{align*}
l_1 : l_2 : l_3 &= q_2 : q_3 : 2q_8 \\
a_0 : a_1 : a_2 &= (q_1^2 + q_2^2 + q_3^2) : (-q_1q_4 - q_5q_6 - 2q_2q_7) : \\
&\quad (-q_1q_4 - q_5q_6 - 2q_2q_7) \\
x_1 : x_2 : x_3 &= (q_6q_4 - 2q_7q_4) : (-q_6q_4 + 2q_7q_4) : (q_5^2 + q_3^2)
\end{align*}
\]

4.6 Sufficiency of the Unified Representation. From Secs. 4.1–4.5, we have found that the algebraic constraints of all the four types of planar dyads, RR, PR, RP and PP, can be converted to a unified representation given by the G-manifold (17) with two fundamental conditions (21). Conversely, it is not difficult to show that when \( q_1 \neq 0 \), the G-manifold (17) whose coefficients satisfy the two conditions (21) reduces to a hyperbolic of one sheet

\[
\begin{align*}
&\left[ q_1Z_1 + \frac{1}{2}(q_2 + q_4)Z_2 + \frac{1}{2}(q_3 - q_5) \right]^2 \\
+ \left[ q_1Z_2 + \frac{1}{2}(q_3 + q_5)Z_3 + \frac{1}{2}(q_2 - q_4) \right]^2 \\
= \frac{1}{4}(q_2^2 + q_4^2 + q_3^2 + q_5^2 - 4q_1q_7)(Z_1^2 + 1)
\end{align*}
\]

and that when \( q_1 = 0 \), the G-manifold satisfying Eq. (21) reduces to a hyperbolic paraboloid

\[
Z_2[(q_3 + q_5)Z_2 + (q_1 + q_7)Z_3 + (q_2 + q_4)Z_1 + q_6] \\
= (q_3 - q_5)Z_3 + (q_2 - q_4)Z_2 + (q_1 - q_7)
\]

Thus, it is concluded that the unified representation is both necessary and sufficient for representing all four types of planar dyads, RR, PR, RP, and PP.

4.7 Unifying Representations for Planar Four-Bar Motions. It is well known that a planar four-bar linkage can be defined by combining two planar dyads from the group of four dyads: RR, PR, RP, and PP. This results in planar 4R, slider–crank, inversions of slider–crank, as well as double slider mechanisms. In the past, the image curve of a planar four-bar linkage has been represented as intersection of two constraint manifolds directly associated with the two dyads. In this paper, however, we represent the image curve by a pencil of quadrics (17) that satisfy the conditions on the coefficients given by Eq. (21). Instead of fitting a pair of constraint manifolds directly, we first fit a pencil of G-manifolds (17) to the set of image points and then impose constraints (21) to identify two C-manifolds from the G-manifolds. This decoupling of constraints (21) from the curve fitting process not only removes the bottleneck in the fitting of the image curve of a four-bar linkage but also unifies the synthesis of all types of planar four-bar linkages. The choice of an \( R \) or \( P \) joint in a four-bar linkage is determined by the input positions only and is obtained after the fitting process for a pencil of G-manifolds.

5 Algebraic Fitting of a Pencil of G-Manifolds

Now consider the problem of fitting a pencil of G-manifolds to a set of \( N \) image points arranged such that they define an image curve rather than a surface. This problem can be formulated as an over-constrained linear problem \([A]q = 0 \) obtained by substituting for the given values of the image points in Eq. (17), where \( q \) is the column vector of homogeneous coefficients \( q_i(i = 1…8) \). The coefficient matrix \([A]\) is given by

\[
[A] = \\
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_{N1} & A_{N2} & A_{N3} & A_{N4} & A_{N5} & A_{N6} & A_{N7} & A_{N8}
\end{bmatrix}
\]

where for the \( i \)th image points, we have

\[
A_{ij} = Z_{ij}^2 + Z_{ij}^3, \quad A_{ij} = Z_{ij}Z_{ij} - Z_{ij}Z_{ij} \\
A_{i1} = Z_{ij}Z_{ij} + Z_{ij}Z_{ij}, \quad A_{i2} = Z_{ij}Z_{ij} + Z_{ij}Z_{ij} \\
A_{ij} = Z_{ij}Z_{ij} - Z_{ij}Z_{ij}, \quad A_{i6} = Z_{ij}Z_{ij} \\
A_{ij} = Z_{ij}Z_{ij} - Z_{ij}Z_{ij}, \quad A_{i8} = Z_{ij}Z_{ij} + Z_{ij}Z_{ij}
\]

5.1 Singular Value Decomposition. In linear algebra, the singular value decomposition (SVD) (see Ref. [30]) of an \( N \times 8 \) matrix \([A]\) is a factorization of the form

\[
[A] = [U][S][V]^T
\]

where \([U]\) is an \( N \times N \) orthonormal matrix, whose \( N \) columns, called the left singular vectors of \([A]\), are the eigenvectors of \([A][A]^T\); \([S]\) is an \( N \times 8 \) rectangular diagonal matrix with eight non-negative real numbers on the diagonal, whose values are square roots of the eigenvalues of \([A][A]^T\) (or equivalently

![A hyperbolic paraboloid defined by \( Z_2Z_2 - Z_2 = 0 \)](image-url)
The over-constrained system of linear equations, \(|A|q = 0\), can be solved as a total least squares minimization problem with the constraint \(q^Tq = 1\). The solution turns out to be the right singular vectors of \(|A|\) corresponding to the least singular values. These vectors form an orthonormal set of basis vectors spanning the null space of \(|A|\), or in other words, solutions to \(|A|q = 0\). Therefore, the rank of matrix \(|A|\), and consequently its nullity (8−rank), will determine the number of zero singular values.

We note that the algebraic distance fitting in SVD uses minimization of the two-norm of the error vector \(|A|q\) over all given image points. Such an expression is unit-inconsistent due to the fact that \(|A|\) contains terms of image space coordinates, which have different dimensions (\(Z_1, Z_2\) have dimensions of translation components, while \(Z_3, Z_4\) are dimensionless). In order to resolve this inconsistency, it is commonly suggested that the translation vector be divided by a characteristic length \(L\) to render the new coordinates dimensionless. Then, the new coordinates are given as

\[Z_1 = \frac{1}{2} \left( d_1 \cos \phi + d_2 \sin \phi \right) \]
\[Z_2 = \frac{1}{2} \left( d_1 \sin \phi - d_2 \cos \phi \right) \]
\[Z_3 = \sin \phi \]
\[Z_4 = \cos \phi \]

(31)

When the translation components are rationalized as in above, we can use two-norm for least-square calculations. The choice of a characteristic length is discussed in detail in Ref. [31]. Another approach is to approximate a planar displacement with a spherical displacement in order to obtain a distance metric that is approximately bi-invariant and unit-consistent. This approach has been discussed in Refs. [32–36], and more recently in Ref. [37]. Purwar and Ge [37] have shown how a planar or spatial displacement can be approximated by a 3D- or four-dimensional-rotation using dual- and double-quaternion approach. Since this paper’s focus is not on distance metric computation, for computational simplicity, we choose \(L = 1\). Any other choice of \(L\) would leave our method unchanged. This is the choice that Ravani and Roth [9] and in recent years Hayes et al. [23,24] have also made although without explicitly mentioning it.

As the matrix \(|A|^T|A|\) is \(8 \times 8\) and positive semidefinite, all eigenvalues are non-negative and the eigenvector associated with the smallest of the eight eigenvalues is a “candidate” solution for \(p = (p_1, p_2, \ldots, p_8)\). When \(n \leq 5\), the matrix \(|A|^T|A|\) has in general (8−n) identical zero eigenvalues, the null space of \(|A|\) is (8−n) dimensional, and is defined by the corresponding orthonormal eigenvectors associated with the zero eigenvalues. Thus, a candidate solution may be expressed as a linear combination of those orthonormal eigenvectors. For \(n \geq 5\), the rank of matrix \(|A|\) is five, then the matrix \(|A|^T|A|\) has three zero-identical eigenvalues and the corresponding eigenvectors, \(v_a, v_b, v_c\), define the basis for the null space. Let \(x, \beta, \gamma\) denote three real homogeneous parameters. Then, any vector in the null space is given by

\[p = xv_a + \beta v_b + \gamma v_c \]

(32)

For vector \(p\) to satisfy Eq. (21), we substitute Eq. (32) into Eq. (21) and obtain two homogeneous quadratic equations in \((x, \beta, \gamma)\)

\[K_{12}x^2 + K_{13}x\beta + K_{14}x\gamma + K_{15}\beta^2 + K_{16}\beta\gamma + K_{17}\gamma^2 = 0 \]
\[K_{23}x^2 + K_{24}x\beta + K_{25}x\gamma + K_{26}\beta^2 + K_{27}\beta\gamma + K_{28}\gamma^2 = 0 \]

(33)

where \(K_{ij}\) are defined by components of the three eigenvectors, which can be obtained from using singular value decomposition of \(|A|\) [30]. As \(x, \beta, \gamma\) are homogeneous, we can set \(\gamma = 1\) in order to solve for \(x\) and \(\beta\).

Solving Eq. (33) and substituting in Eq. (32) would lead to the homogeneous coordinates of dyads. For a set of \(n\) task positions, the aforementioned task analysis algorithm may yield up to four dyads from the solution of two quadratic equations in Eq. (21), two of which can be combined to form up to six four-bar linkages. Design parameters such as \(x_1, y_1, z_1\) and \((a_0, a_1, a_2, a_3)\) can be obtained from inverse relationships given in Eq. (25).

In short, this approach leads to a unified algorithm for both exact synthesis (when \(n \leq 5\)) and approximate synthesis (when \(n > 5\)) of planar dyads that can handle joint type and dimensional synthesis simultaneously.

6 Examples

Now, we present three examples that illustrate our approach. These examples do not presume the linkage type and determine the best types and dimensions from the given motion.

6.1 Example: Motion of an Aircraft Landing Gear. Five positions for the landing gear of an aircraft are shown in Fig. 5 and listed in Table 1. The objective is to find a four-bar mechanism, which can realize this motion.

The first step of the two-step algorithm is to extract geometric constraints of motion and fit a pencil of G-manifolds to it. This is done by creating matrix \(|A|\) using Eq. (28) and applying SVD to it. Since the nullity of \(|A|\) is 3, we pick three singular vectors associated with near-zero singular values. Singular values and singular vectors are presented in Tables 2 and 3. These singular vectors form a pencil of G-manifolds defined by Eq. (32). Table 3 also contains constraint fitting error for each of the singular vectors, clearly indicating that none of them correspond to any type of mechanical dyad.

![Fig. 5 Example 6.1: five positions of an aircraft landing gear labeled 1 ... 5 are shown. The moving frame is attached to the top left corner of the housing, while frame XY is the fixed frame.](image-url)
The second step is to impose constraints (33) to identify C-manifolds from the pencil of G-manifolds. Solving these equations leads to two real solutions of \( x, \beta \) given by

\[
\begin{align*}
    x_1 &= -1.9303, \quad \beta_1 = 0.2417 \\
    x_2 &= 1.4469, \quad \beta_2 = 0.6985
\end{align*}
\]  

(34)

Each real solution forms a C-manifold; see Table 4. It can be seen in this table that fitting error is of the order of \( 10^{-17} \); hence, these manifolds are constraint manifolds of planar dyads. By examining the coefficients of vectors \( p_1 \) and \( p_2 \), it can be easily shown that first vector corresponds to the constraint manifold of an \( RR \) dyad (\( q_1 \neq 0 \)), whereas the second vector corresponds to the constraint manifold of a \( PR \) dyad (\( q_1 = q_2 = q_3 = 0 \)). Hence, a slider crank mechanism is formed by joining these two dyads via coupler as shown in Fig. 5. The C-manifolds projected on hyperplane \( \mathbb{Z}_4 = 1 \) are shown in Fig. 6. The linkage parameters can be obtained from inverse kinematic equations (25). Parameters for dyad 1 are: \( a_0: a_1: a_2: a_3 = 5.9352 \times 10^{-3}: 0.0387: 0.05989: -109.8479, x_1: x_2: x_3 = 7.1373: -2.3250: 1 \), while for dyad 2, they are: \( a_0: a_1: a_2: a_3 = 4.7822 \times 10^{-10}: 1.4528 \times 10^{-7}: 1.4365 \times 10^{-5}: -328, 400.709, x_1: x_2: x_3 = 2.8282: 3.7737: 1 \).

6.2 Example: ASME Mechanism Design Challenge. McCarthy at the 2002 ASME IDETC [38] proposed a mechanism design challenge where the objective was to synthesize a four-bar linkage to follow a motion defined by 11 poses as shown in Fig. 7. In general, such a motion can be only approximated by a four-bar linkage.

Applying our algorithm, we obtain the singular values and singular vectors as listed in Tables 5 and 6, respectively. The singular vectors form the basis for a pencil of G-manifolds. Then, we impose constraints (33) to identify C-manifolds from the pencil of G-manifolds. Solving equations (33) leads to two real solutions of \( x \) and \( \beta \) given by

\[
\begin{align*}
    x_1 &= 1.4526, \quad \beta_1 = -0.5846 \\
    x_2 &= 1.2283, \quad \beta_2 = 1.2944
\end{align*}
\]  

(35)

Each real solution forms a C-manifold. Table 7 contains vector coefficients corresponding to C-manifolds obtained. The projection of these C-manifolds on hyperplane \( \mathbb{Z}_4 = 1 \) is depicted in Fig. 8. It can be seen in Table 7 that fitting error is of the order of \( 10^{-11} \); hence, these manifolds are constraint manifolds of planar

![Fig. 6 Example 6.1: two resulting constraint manifolds identified from a pencil of G-manifolds that satisfy Eq. (21) are illustrated in this figure by projecting them on hyperplane \( \mathbb{Z}_4 = 1 \). Intersection of hyperboloid and hyperbolic paraboloid forms constraint manifold of the slider crank mechanism. Five black image points on the intersection curve show projection of five task positions on hyperplane \( \mathbb{Z}_4 = 1 \).](https://asmedigitalcollection.asme.org/computingengineering/article-pdf/17/3/031011/5997059/jcise_017_03_031011.pdf)

![Fig. 7 Example 6.2: 11 task positions of ASME Mechanism Design Challenge and its solution as synthesized four-bar mechanism](https://asmedigitalcollection.asme.org/computingengineering/article-pdf/17/3/031011/5997059/jcise_017_03_031011.pdf)
easily concluded that each of the dyad corresponds to an RR dyad, which forms the four-bar mechanism depicted in Fig. 7. It is seen from a known four-bar linkage. Using inverse kinematic equations (25), we obtain linkage parameters. The parameters for dyad 1 are: $a_0: a_1: a_2: a_3 = 0.2146: 0.4728: 0.3483: -4.3961, x_1: x_2: x_3 = 1.9487: 1.3921: 1$, and for dyad 2 they are: $a_0: a_1: a_2: a_3 = 0.5942: 0.4726: 0.2169: 2.2744, x_1: x_2: x_3 = 0.0615: 1.5700: 1$.

6.3 Example: Sit-to-Stand (STS) Motion. Now, we present an example where a linkage that can execute a sit-to-stand (STS) motion for people suffering from neuromuscular disabilities is to be synthesized. It is desirable that the orientation of the upper body to which the coupler will be attached remains constant during the STS motion. We specify five task positions with the same orientation, i.e., 0° but the coupler goes through five different locations as shown in Table 8.

Since all five task positions share the same orientation and the positions are not on a circle, it is known that no four-bar linkage could realize this motion. So, we try to find a six bar which can realize this motion. There are many ways in which different types of six bar mechanisms can be synthesized. Here we employ Soh and McCarthy’s strategy of synthesizing six-bar linkages [1], which is to start with finding a 3R triad to realize the task positions (Fig. 9), and then form a 1DOF closed-chain six-bar linkage by adding additional links. The process of adding a new link requires the synthesis of dyads, which are obtained by repeated application of our algorithm. For a triad shown in Fig. 9, we pick the location of fixed pivot (F1) and third joint (L1) as well as the length of the two links (links 2 and 3) between them. The given task positions are located at L1. Although there are an infinite number of triads that can reach these task positions, we pick length for links 2 and 3 to be 10.5 and 14.92, respectively, such that the triad’s workspace contains the given positions; see Ref. [21] for an image-based graphical approach to selecting planar triads. Next, we use inverse kinematics to obtain the locations of joint M1 and orientations of links 2 and 3 at various task positions. Table 9 contains orientations of link 2 corresponding to each task position. Next, we form links 4 (L1L2) and 5 (L2M3) to create a two-DOF five-bar linkage; see Fig. 10. These link lengths can be

<table>
<thead>
<tr>
<th>Table 5</th>
<th>Example 6.2: singular values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$-4.173 	imes 10^{-4}$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$-1.725 	imes 10^{-5}$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$0.5653$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 6</th>
<th>Example 6.2: singular vectors obtained from singular value decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$0.17676$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$0.41615$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 7</th>
<th>Example 6.2: C-manifolds corresponding to RR dyads of four-bar synthesized as shown in Fig. 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$0.17676$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$0.41615$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 8</th>
<th>Example 6.3: five positions for sit to stand motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position</td>
<td>$x$</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>-6.41</td>
</tr>
<tr>
<td>2</td>
<td>-3.85</td>
</tr>
<tr>
<td>3</td>
<td>-0.40</td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
</tr>
<tr>
<td>5</td>
<td>0.90</td>
</tr>
</tbody>
</table>

![Fig. 8 Example 6.2: projection of C-manifolds tabulated in Table 7. Their intersection forms constraint manifold of the four-bar linkage. Eleven black image points on the intersection curve represent projection of task positions on hyperplane $Z_1 = 1$.](image1)

![Fig. 9 Example 6.3: the synthesis starts by finding a 3R triad whose workspace contains the given task positions. The figure shows triad at the first task position.](image2)
Fig. 10  Example 6.3: the figure shows a five-bar linkage, where links 4 and 5 are synthesized relative to the link 2. Figure shows locations of fixed and moving pivots of chosen dyad obtained as a result of relative synthesis.

Table 9  Example 6.3: orientations of link 2 after inverse kinematics computation

<table>
<thead>
<tr>
<th>Task position</th>
<th>Orientation of link 2 (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>139.69</td>
</tr>
<tr>
<td>2</td>
<td>134.49</td>
</tr>
<tr>
<td>3</td>
<td>101.05</td>
</tr>
<tr>
<td>4</td>
<td>64.95</td>
</tr>
<tr>
<td>5</td>
<td>68.16</td>
</tr>
</tbody>
</table>

Table 10  Example 6.3: singular values obtained in relative synthesis of links 4 and 5

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>22.209</td>
<td>92.271</td>
<td>15.0021</td>
<td>0.23189</td>
<td>0.01793</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7.7724×10⁻¹²</td>
</tr>
</tbody>
</table>

Table 11  Example 6.3: singular vectors associated with near-zero singular values presented in Table 10

<table>
<thead>
<tr>
<th>Vector</th>
<th>q₁</th>
<th>q₂</th>
<th>q₃</th>
<th>q₄</th>
<th>q₅</th>
<th>q₆</th>
<th>q₇</th>
<th>q₈</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>v₁</td>
<td>-0.0399</td>
<td>0.0343</td>
<td>0.0120</td>
<td>0.4074</td>
<td>0.0847</td>
<td>-0.9018</td>
<td>-0.1025</td>
<td>0.0121</td>
<td>0.1564</td>
</tr>
<tr>
<td>v₂</td>
<td>-0.0464</td>
<td>0.1559</td>
<td>-0.1646</td>
<td>0.3061</td>
<td>0.0742</td>
<td>0.0465</td>
<td>0.9193</td>
<td>0.0123</td>
<td>0.0574</td>
</tr>
<tr>
<td>v₃</td>
<td>-0.0229</td>
<td>-0.0222</td>
<td>0.0164</td>
<td>0.2623</td>
<td>-0.0206</td>
<td>0.1268</td>
<td>-0.08658</td>
<td>0.9517</td>
<td>0.1463</td>
</tr>
</tbody>
</table>

Table 12  Example 6.3: C-manifolds obtained in relative synthesis of links 4 and 5

<table>
<thead>
<tr>
<th>Vector</th>
<th>q₁</th>
<th>q₂</th>
<th>q₃</th>
<th>q₄</th>
<th>q₅</th>
<th>q₆</th>
<th>q₇</th>
<th>q₈</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>p₁</td>
<td>7.15×10⁻³</td>
<td>4.26×10⁻³</td>
<td>-0.0733</td>
<td>-0.1191</td>
<td>-0.0952</td>
<td>1.2988</td>
<td>0.4599</td>
<td>0.9518</td>
<td>4.65×10⁻¹³</td>
</tr>
<tr>
<td>p₂</td>
<td>0.0115</td>
<td>-0.1558</td>
<td>0.1668</td>
<td>0.0615</td>
<td>-0.0718</td>
<td>-0.0805</td>
<td>-0.9326</td>
<td>0.9517</td>
<td>7.69×10⁻¹⁴</td>
</tr>
<tr>
<td>p₃</td>
<td>-3.14×10⁻³</td>
<td>-0.0265</td>
<td>-9.22×10⁻³</td>
<td>0.0411</td>
<td>-0.0654</td>
<td>0.6965</td>
<td>0.0791</td>
<td>0.9517</td>
<td>6.54×10⁻¹⁴</td>
</tr>
<tr>
<td>p₄</td>
<td>-0.0334</td>
<td>-4.40×10⁻¹³</td>
<td>-1.63×10⁻¹³</td>
<td>0.3557</td>
<td>-4.36×10⁻¹³</td>
<td>1.72×10⁻¹²</td>
<td>2.31×10⁻¹²</td>
<td>0.9517</td>
<td>2.35×10⁻¹³</td>
</tr>
</tbody>
</table>

Table 13  Example 6.3: singular values obtained in synthesis of link 6

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>12,164</td>
<td>231.45</td>
<td>24.2207</td>
<td>6.0531</td>
<td>2.0841</td>
<td>7.7724×10⁻¹²</td>
<td>4.1753×10⁻¹³</td>
<td>2.4375×10⁻¹³</td>
</tr>
</tbody>
</table>

Table 14  Example 6.3: singular vectors associated with zero singular values presented in Table 13

<table>
<thead>
<tr>
<th>Vector</th>
<th>q₁</th>
<th>q₂</th>
<th>q₃</th>
<th>q₄</th>
<th>q₅</th>
<th>q₆</th>
<th>q₇</th>
<th>q₈</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>v₁</td>
<td>-0.0137</td>
<td>-1.14×10⁻³</td>
<td>-4.63×10⁻³</td>
<td>0.1287</td>
<td>-0.062×10⁻³</td>
<td>-6.04×10⁻³</td>
<td>-0.0458</td>
<td>0.9885</td>
<td>0.2416</td>
</tr>
<tr>
<td>v₂</td>
<td>0.0161</td>
<td>0.0653</td>
<td>0.0148</td>
<td>0.02145</td>
<td>-0.1055</td>
<td>0.9896</td>
<td>0.0650</td>
<td>0.0123</td>
<td>0.1432</td>
</tr>
<tr>
<td>v₃</td>
<td>-0.0191</td>
<td>-0.0103</td>
<td>-0.0934</td>
<td>-0.3013</td>
<td>0.0750</td>
<td>0.0788</td>
<td>-0.9424</td>
<td>0.4363</td>
<td>0.1624</td>
</tr>
</tbody>
</table>

Next step of the process is to impose constraints (33) to identify C-manifolds from the pencil of G-manifolds. Solving these equations leads to four real solutions of x, β given by

\[
\begin{align*}
\alpha_1 &= -1.2761, & \beta_1 &= 0.4521 \\
\alpha_2 &= 0.1834, & \beta_2 &= -0.8998 \\
\alpha_3 &= -0.6259, & \beta_3 &= 0.1104 \\
\alpha_4 &= 0.1463, & \beta_4 &= 0.1105
\end{align*}
\]
Table 16  Example 6.3: location for pivots when mechanism is at the first pose

<table>
<thead>
<tr>
<th>Name of pivot</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>11.329</td>
<td>-5.283</td>
</tr>
<tr>
<td>M1</td>
<td>3.321</td>
<td>1.509</td>
</tr>
<tr>
<td>L1</td>
<td>-6.410</td>
<td>-9.800</td>
</tr>
<tr>
<td>M3</td>
<td>-10.309</td>
<td>-4.651</td>
</tr>
<tr>
<td>L2</td>
<td>-7.196</td>
<td>0.601</td>
</tr>
<tr>
<td>M2</td>
<td>-9.308</td>
<td>0.908</td>
</tr>
<tr>
<td>F2</td>
<td>-9.190</td>
<td>11.341</td>
</tr>
</tbody>
</table>

Fig. 11  Example 6.3: the synthesized six-bar linkage for the generation of sit-to-stand motion at the first task position. Image at top right corner shows curve traced by shoulder joint of the human skeleton performing sit to stand motion.

7 Conclusions

In this paper, we presented a novel method for synthesizing planar motion using kinematic mapping. Instead of finding two special quadric constraint manifolds associated with a four-bar linkage with nonlinear (quadratic) coefficients, which makes the problem difficult to solve, we used a more general form of quadric such that its coefficients are linear. Furthermore, we seek to fit a given set of image points to a pencil of quadrics. This leads to a linear least squares problem that can be readily solved using SVD algorithm.

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